CS 6511: Artificial Intelligence

Uncertainty and Utilities

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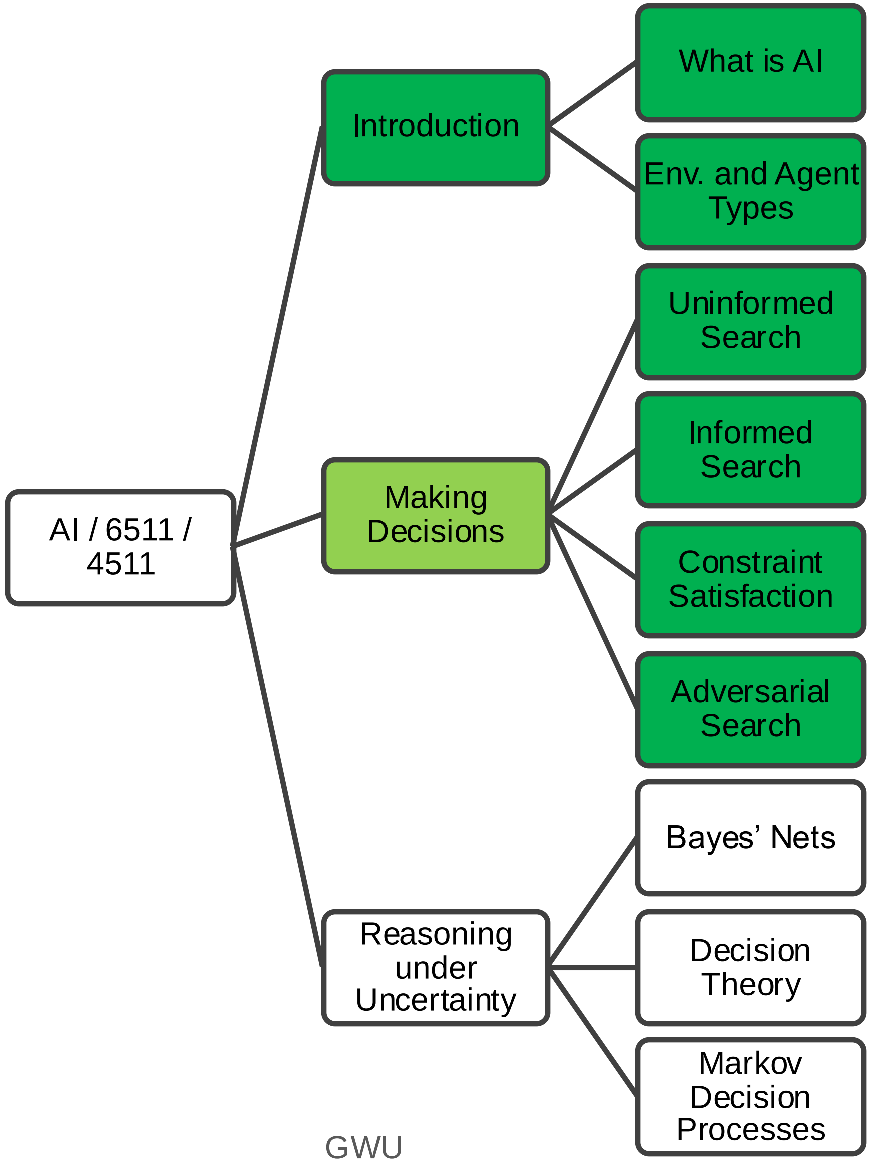
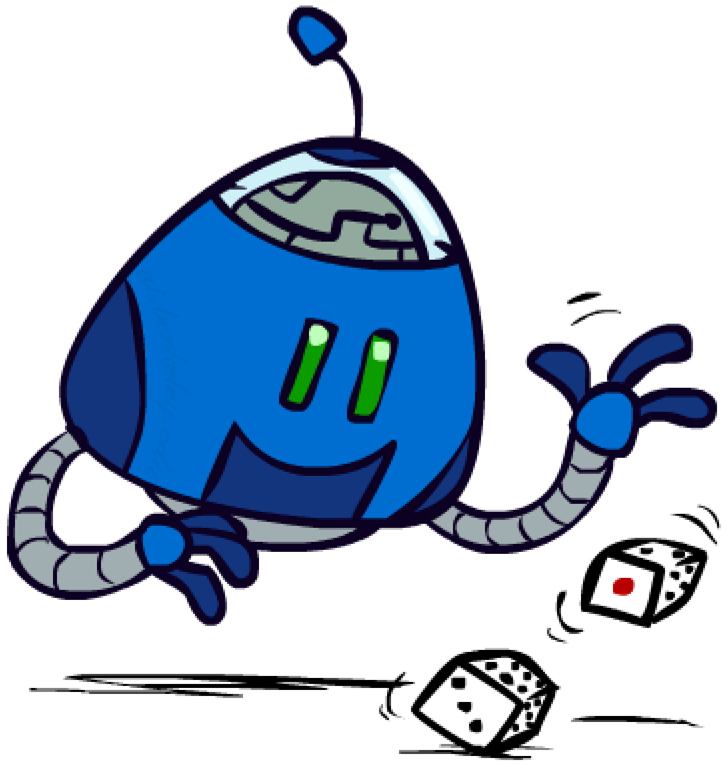
[Original version of these slides was created by Dan Klein and Pieter Abbeel for Intro to AI at UC Berkeley. http://ai.berkeley.edu]

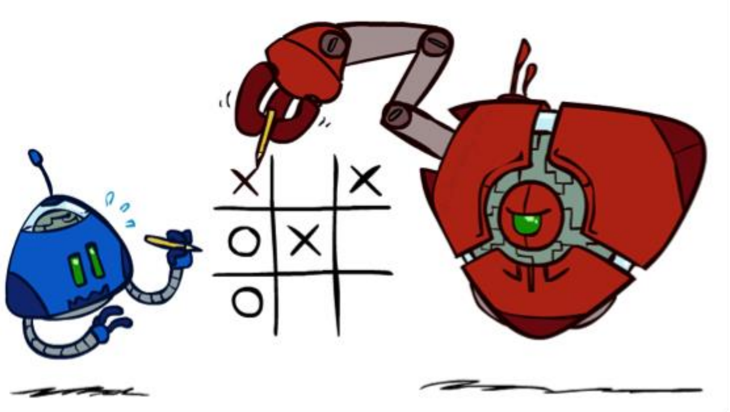
Course Outline

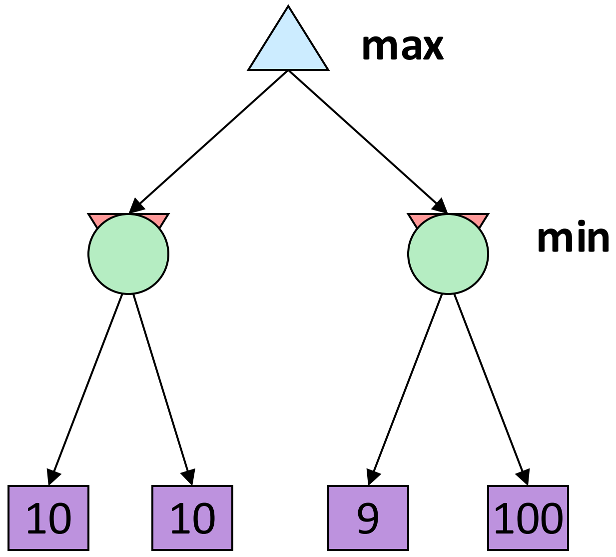


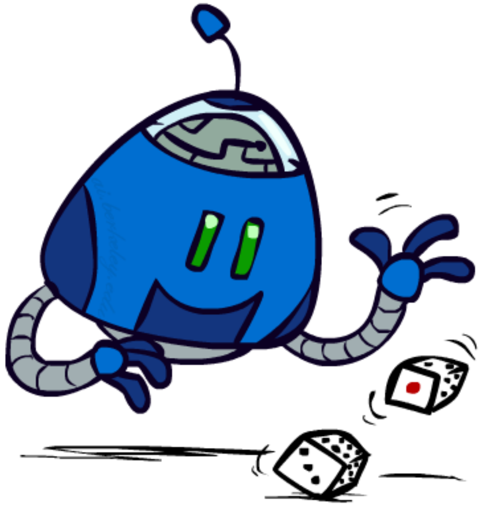
Outcomes Based on Chance



Worst-Case vs. Average Case



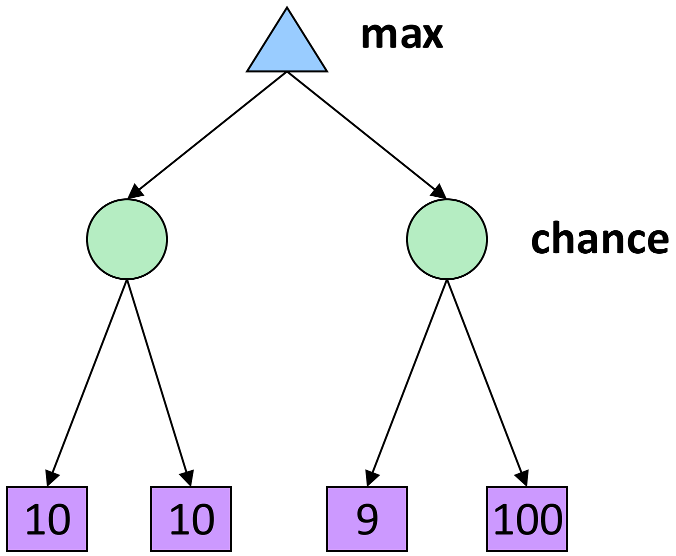




Idea: Uncertain outcomes controlled by chance, not (just) by an adversary!

Expectimax Search

- Why wouldn't we know what the result of an action will be?
  - Explicit randomness: rolling dice
  - Unpredictable opponents: the ghosts respond randomly
  - Actions can fail: when moving a robot, wheels might slip
- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes
- Expectimax search: compute the average score under optimal play
  - Max nodes as in minimax search
  - Chance nodes are like min nodes but the outcome is uncertain
  - Calculate their expected utilities
  - I.e. take weighted average (expectation) of children
- Later, we'll learn how to formalize the underlying uncertain-result problems as Markov Decision Processes



Expectimax Pseudocode

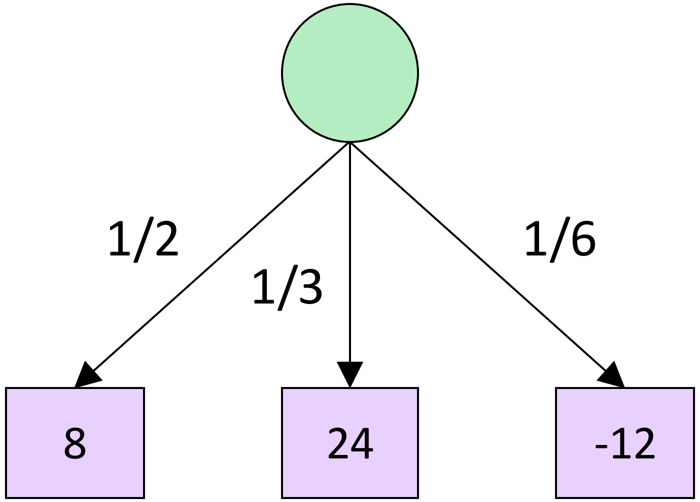
```
def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)
```

```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

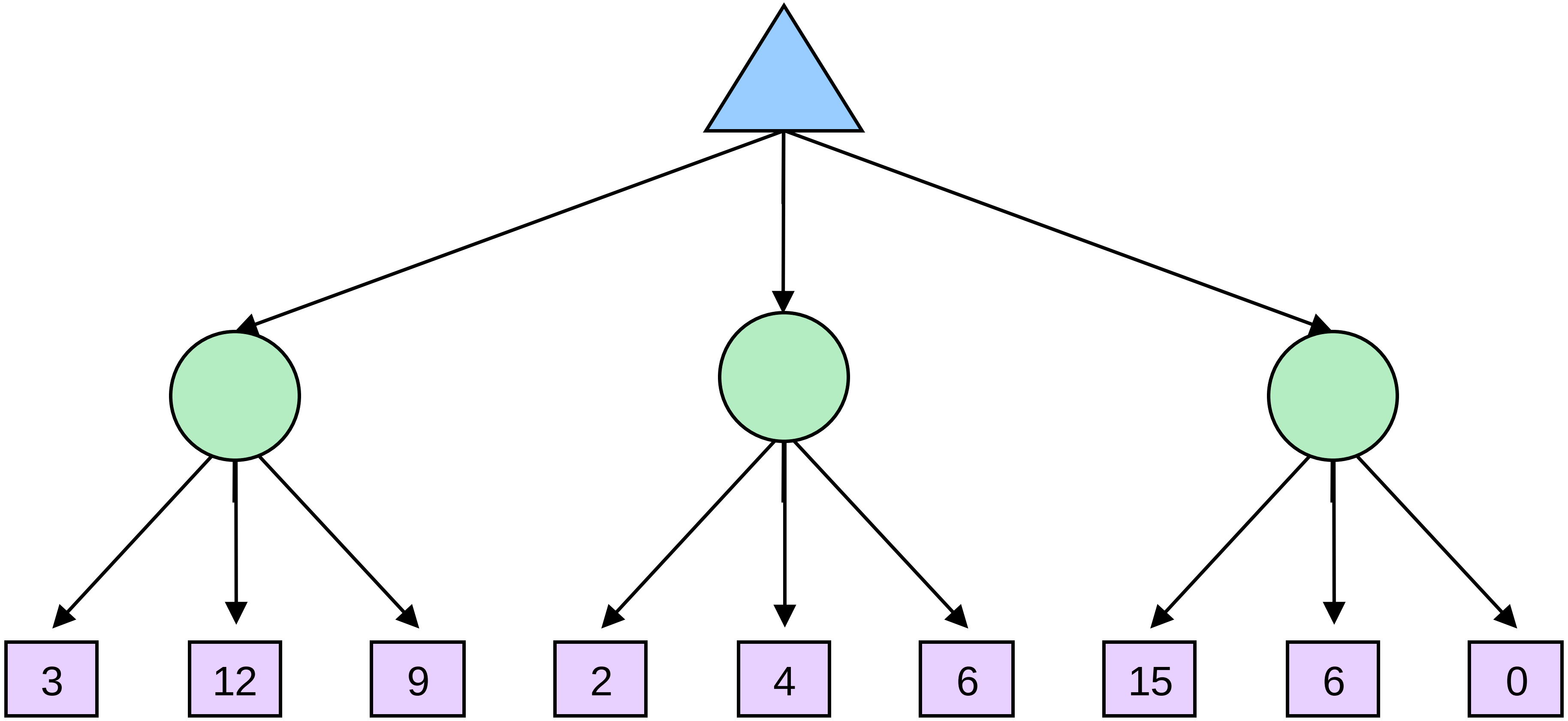
Expectimax Pseudocode

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

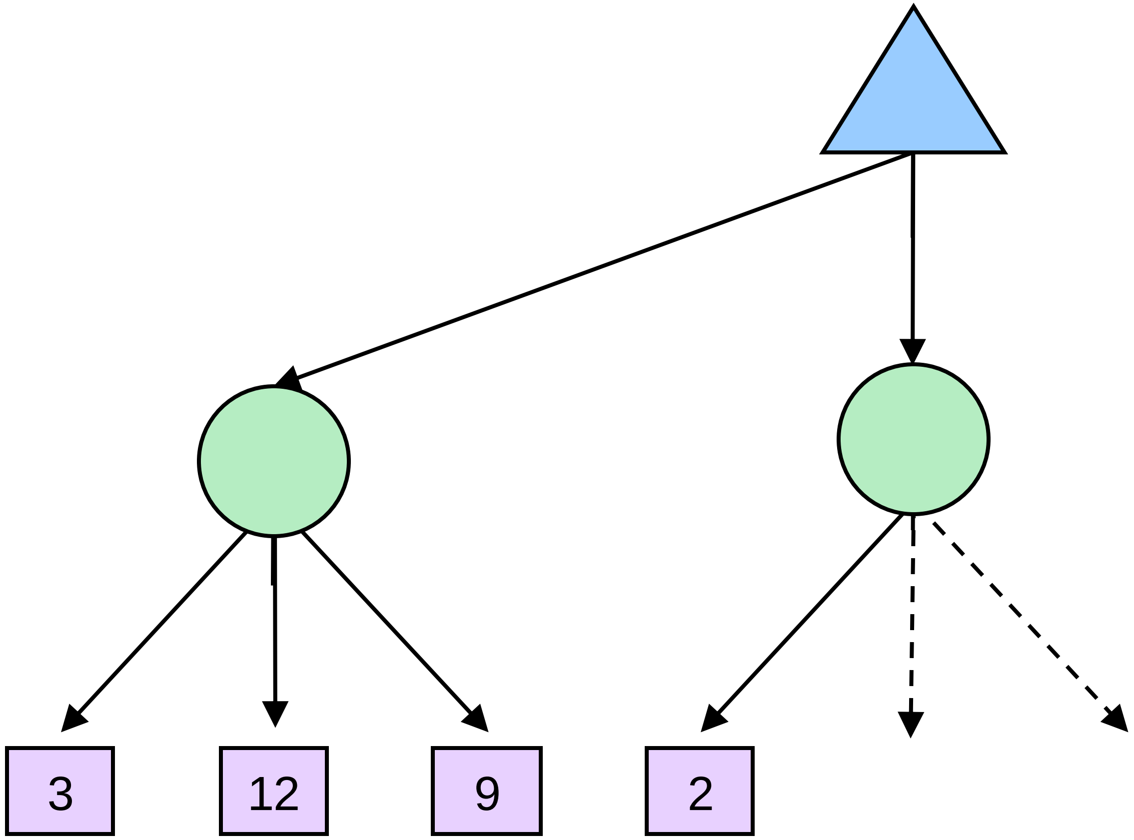


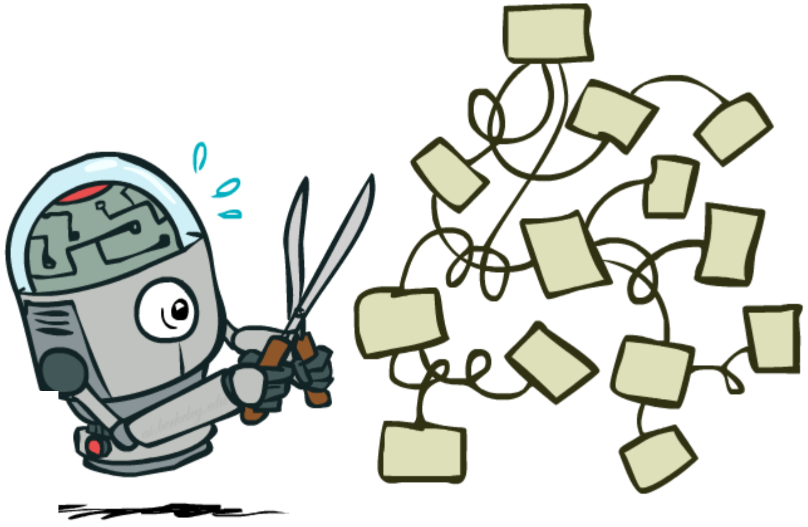
v = (1/2) (8) + (1/3) (24) + (1/6) (-12) = 10

Expectimax Example

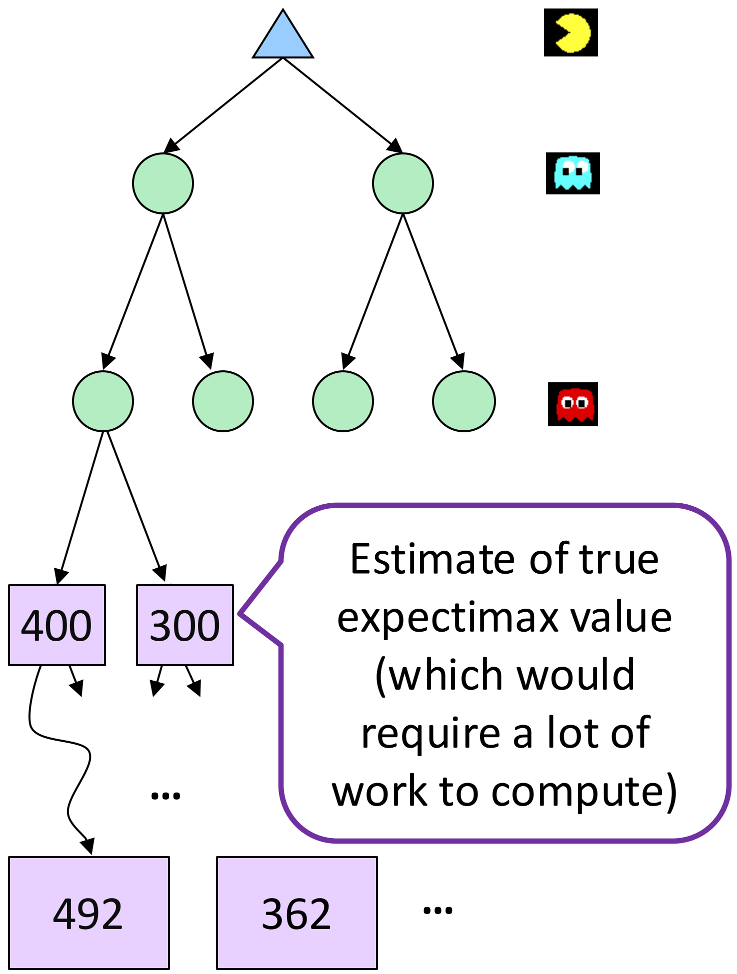


Expectimax Pruning?

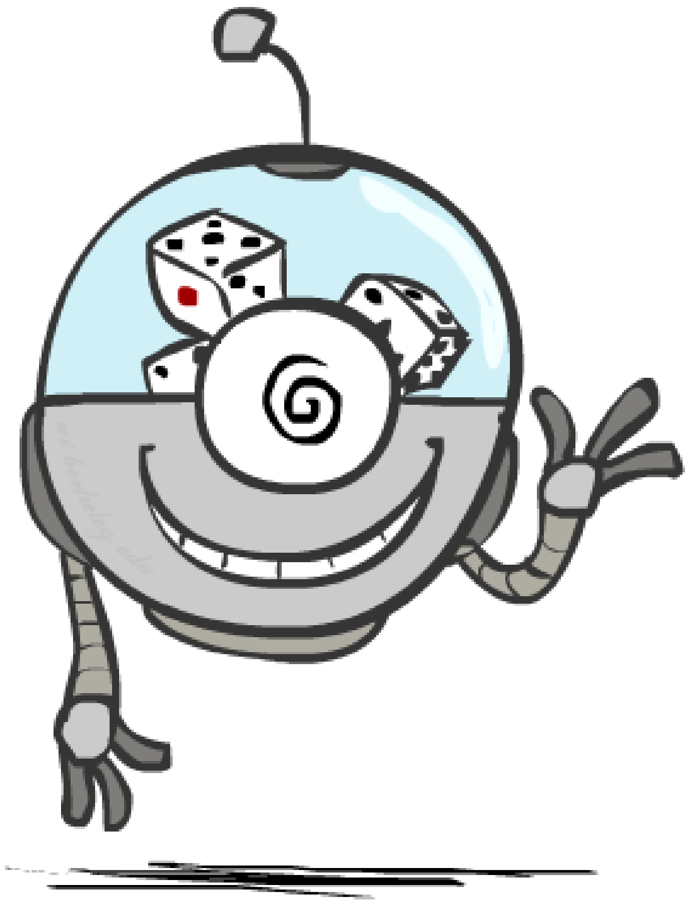




Depth-Limited Expectimax



Probabilities



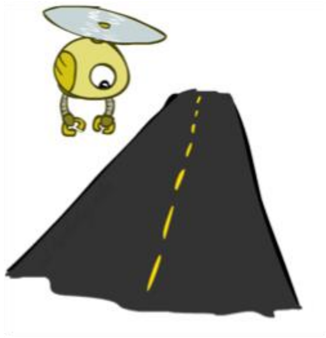
Reminder: Probabilities

- A random variable represents an event whose outcome is unknown
- A probability distribution is an assignment of weights to outcomes

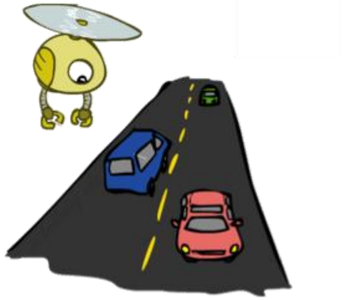
- Example: Traffic on freeway
  - Random variable: T = whether there's traffic
  - Outcomes: T in {none, light, heavy}
  - Distribution: P(T=none) = 0.25, P(T=light) = 0.50, P(T=heavy) = 0.25

- Some laws of probability (more later):
  - Probabilities are always non-negative
  - Probabilities over all possible outcomes sum to one

- As we get more evidence, probabilities may change:
  - P(T=heavy) = 0.25, P(T=heavy | Hour=8am) = 0.60
  - We'll talk about methods for reasoning and updating probabilities later



0.25



0.50

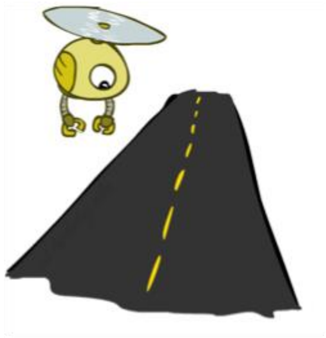
0.25

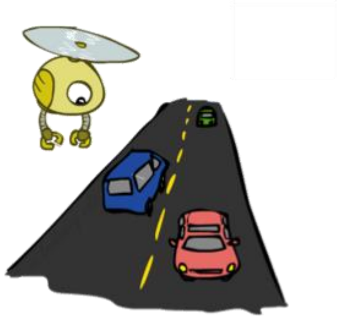
Reminder: Expectations

- The expected value of a function of a random variable is the average, weighted by the probability distribution over outcomes

- Example: How long to get to the airport?

Time:	20 min		30 min		60 min		
	x	+	x	+	x	⇒	35 min
Probability:	0.25		0.50		0.25		





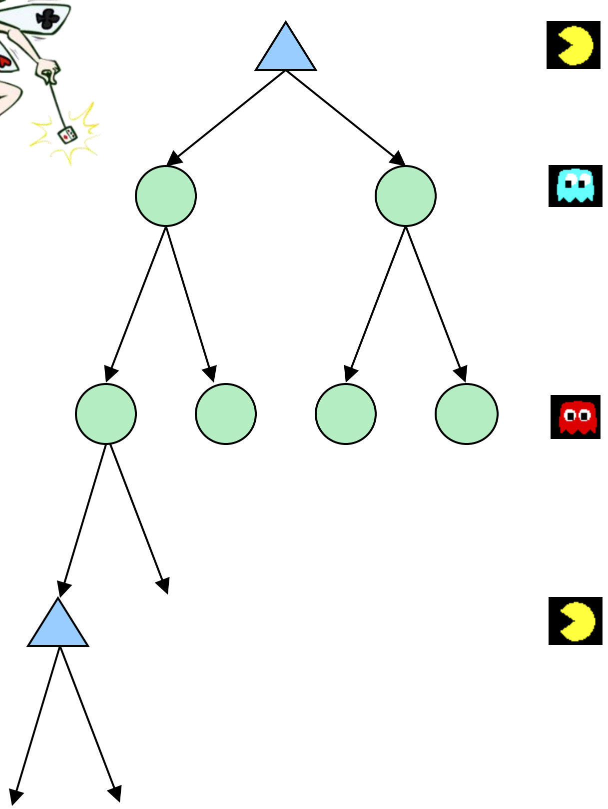
Expected Value Computation Quiz

- $P(X = m) = 1/2^m$ // For all $m \geq 1$
- Clearly, $\Sigma\ 1_{1 <= m <= infinity}(P(X = m)) = 1$
- $E(X) = ?$

What Probabilities to Use?

- In expectimax search, we have a probabilistic model of how the opponent (or environment) will behave in any state
  - Model could be a simple uniform distribution (roll a die)
  - Model could be sophisticated and require a great deal of computation
  - We have a chance node for any outcome out of our control: opponent or environment
  - The model might say that adversarial actions are likely!

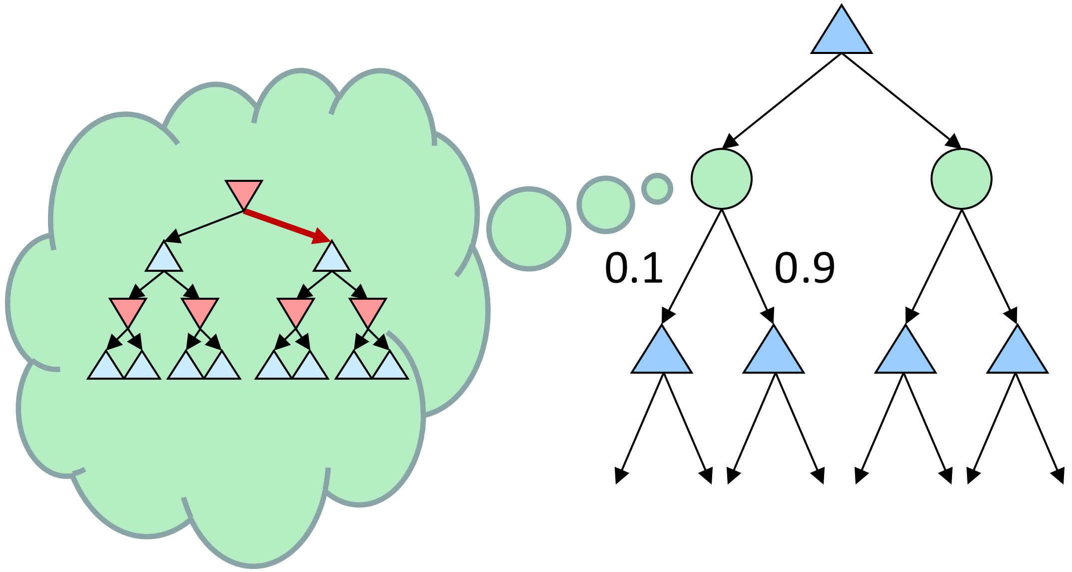
- For now, assume each chance node magically comes along with probabilities that specify the distribution over its outcomes



Having a probabilistic belief about another agent's action does not mean that the agent is flipping any coins!

Informed Probabilities

- Let's say you know that your opponent is actually running a depth 2 minimax, using the result 80% of the time, and moving randomly otherwise
- What tree search should you use?

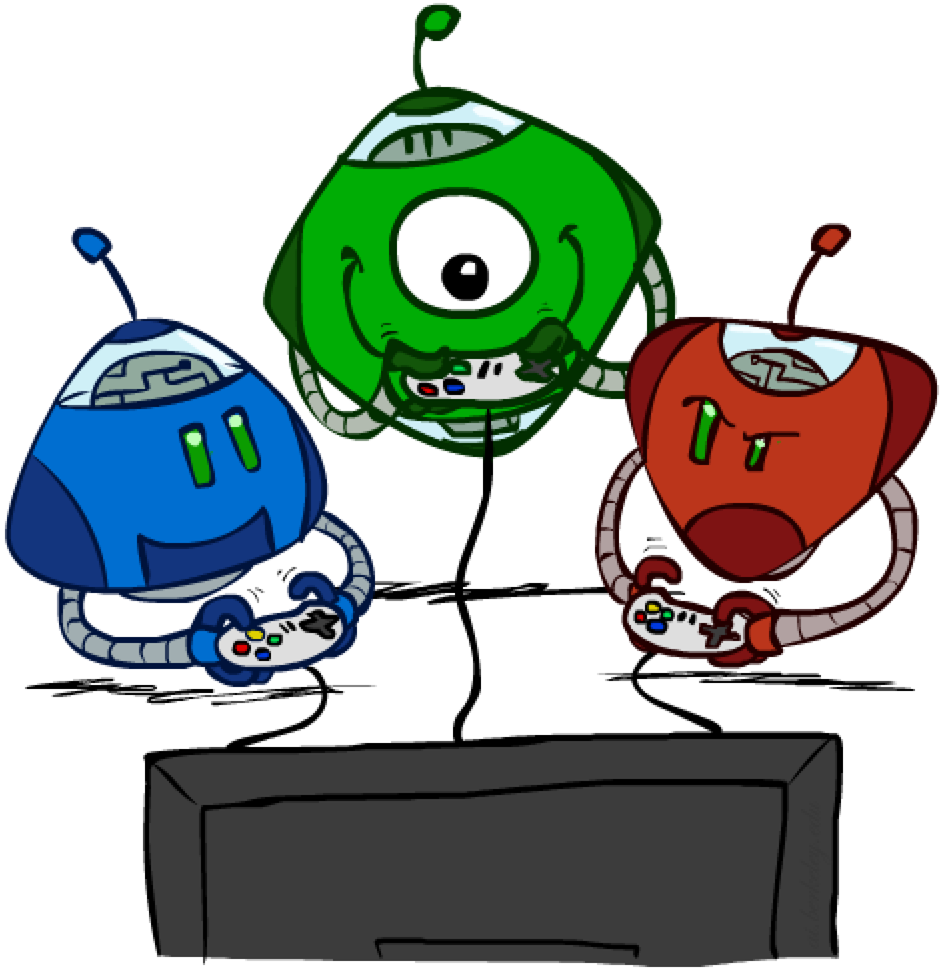


- Answer: Expectimax!
  - To figure out EACH chance node's probabilities, you have to run a simulation of your opponent
  - This kind of thing gets very slow very quickly
  - Even worse if you have to simulate your opponent simulating you…
  - … except for minimax, which has the nice property that it all collapses into one game tree

Other Related Topics

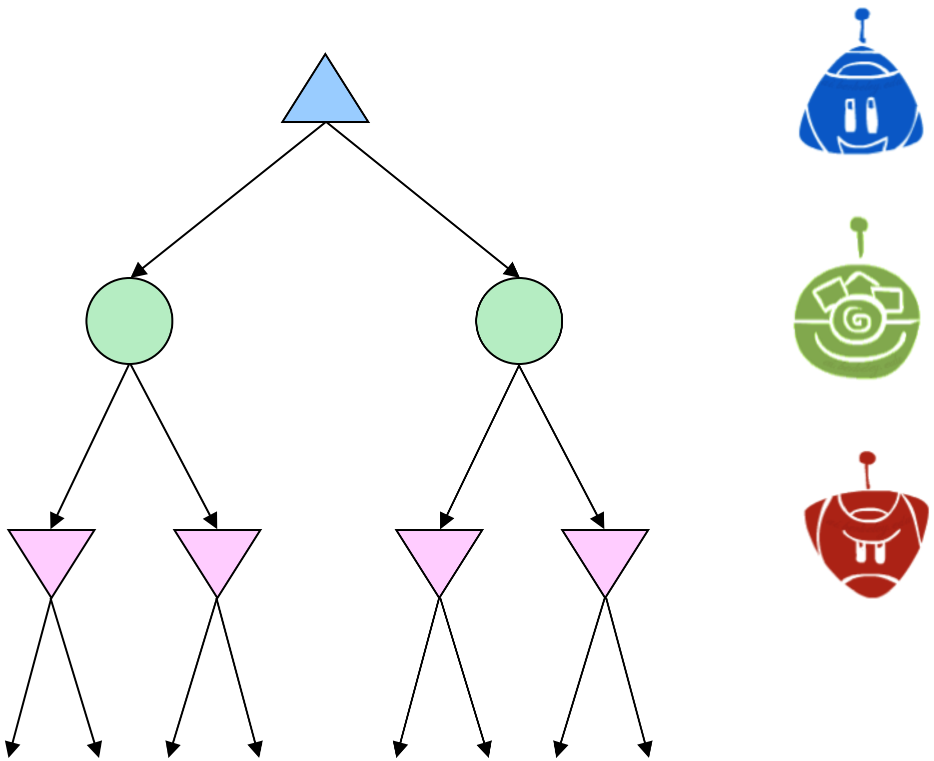
How to handle:

- Mixed layer types
- Mixed agent utilities for a multi agent game



Mixed Layer Types

- For example: backgammon
- Expectiminimax
  - Environment is an extra "random agent" player that moves after each min/max agent
  - Each node computes the appropriate combination of its children

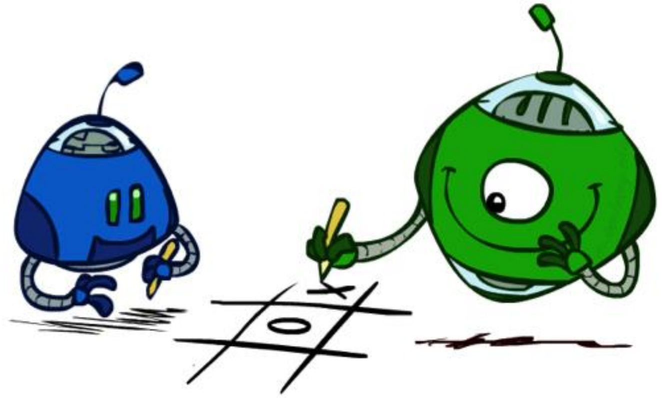


Example: Backgammon

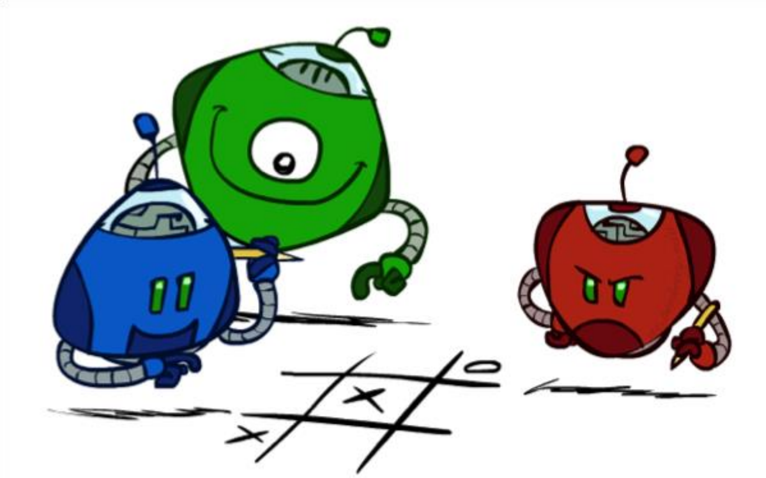
- Dice rolls increase *b*: 21 possible rolls with 2 dice
  - Backgammon ≈ 20 legal moves
  - Depth 2 = 20 x (21 x 20)^3 = 1.2 x 10^9
- As depth increases, probability of reaching a given search node shrinks
  - So usefulness of search is diminished
  - So limiting depth is less damaging
  - But pruning is trickier…
- Historic AI: TDGammon uses depth-2 search + very good evaluation function + reinforcement learning: world-champion level play
- 1st AI world champion in any game!

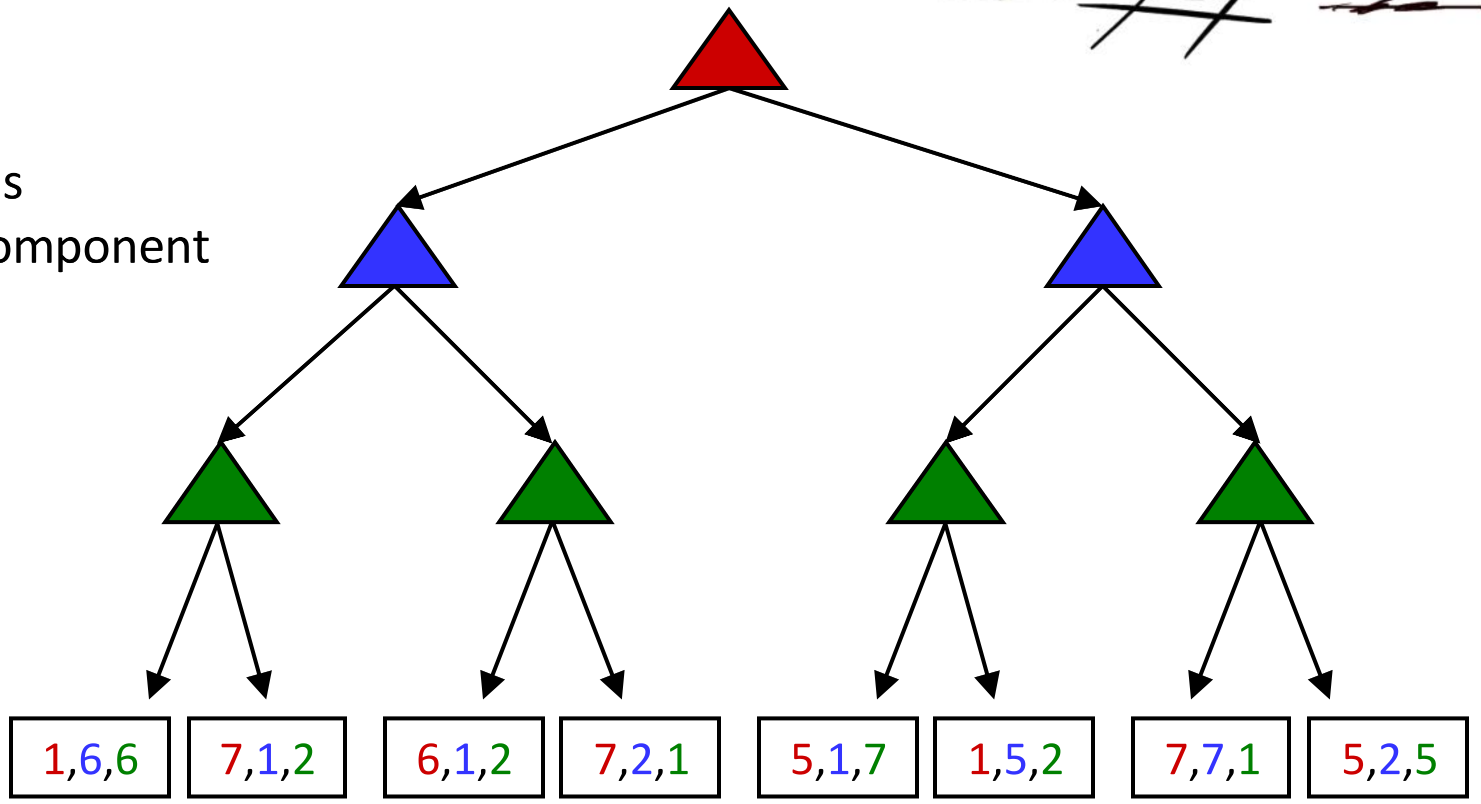
Image: Wikipedia

Multi-Agent Utilities



- What if the game is not zero-sum, or has multiple players?
- Generalization of minimax:
  - Terminals have utility tuples
  - Node values are also utility tuples
  - Each player maximizes its own component
  - Can give rise to cooperation and competition dynamically…





UTILITIES

Maximum Expected Utility

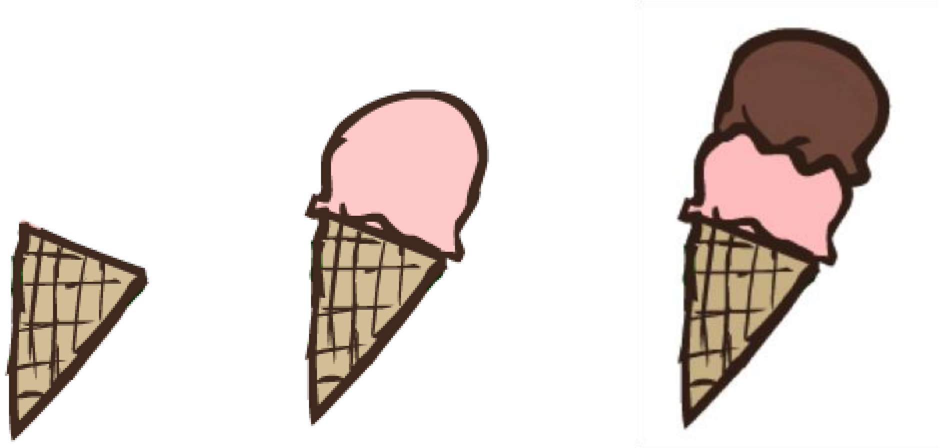
- Principle of maximum expected utility:
  - A rational agent should choose the action that maximizes its expected utility, given its knowledge

- Things to consider:
  - So far, we know “outcomes” and probabilities”, so how should we define this concept of “utility”?
  - How do we know such utilities even exist?
  - How do we know that averaging (**“expected”** utility) even makes sense?
  - What if our behavior (preferences) can’t be described by utilities?

Utilities

- Utilities are functions from outcomes (states of the world) to real numbers that describe an agent's preferences

- Where do utilities come from?
  - In a game, may be simple (+1/-1)
  - Utilities summarize the agent's goals
  - Theorem: any "rational" preferences can be summarized as a utility function



- We hard-wire utilities and let behaviors emerge
  - Why don't we let agents pick utilities?
  - Why don't we prescribe behaviors?

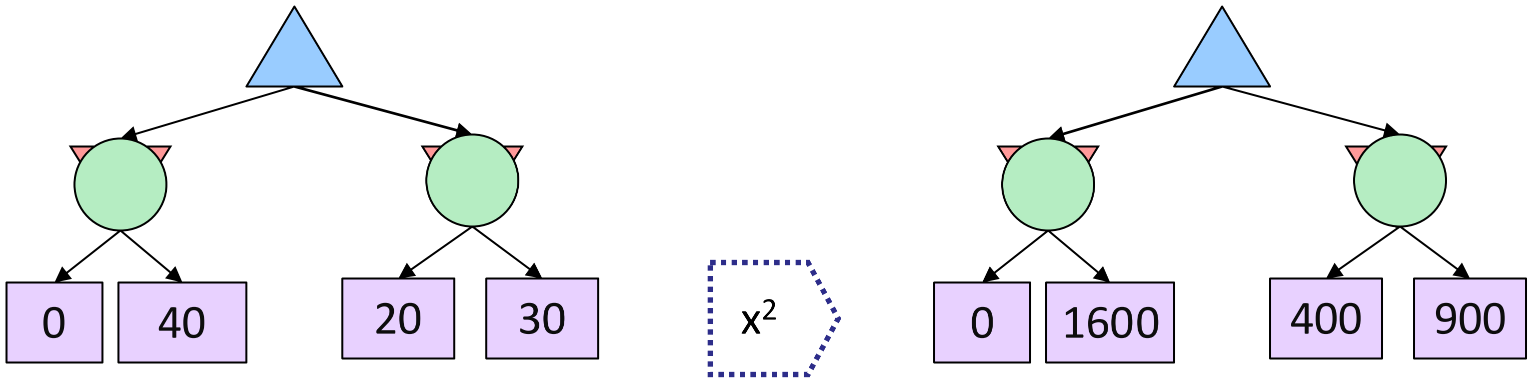
What Utilities to Use?

- For worst-case minimax reasoning, terminal function scale doesn't matter
  - We just want better states to have higher evaluations (get the ordering right)
  - We call this insensitivity to monotonic transformations

- For average-case expectimax reasoning, we need *magnitudes* to be meaningful

- This is a difference that we now need to understand and appreciate.

What Utilities to Use?



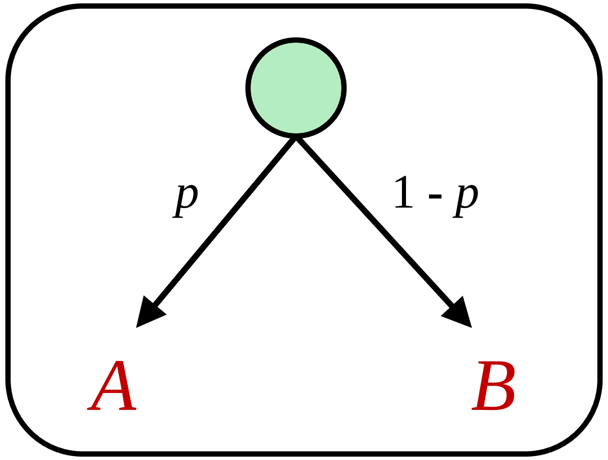
- Utility Values 1 and Values 2 would give us the same answer if using Minimax.
- But, give us different answers when using Expectimax.
- Although, the transformation is monotonic.

Outcomes: Prizes and Lotteries

A Prize

A

A Lottery



$$L = [p, A;\ \ (1-p), B]$$

Preferences

- An agent must have preferences among:
  - Prizes: A, B, etc.
  - Lotteries: situations with uncertain prizes

    $L_1 = [p, A; (1\text{-}p), B]$

    $L_2 = [0.2, A; 0.5, B; 0.3, L_1]$

- Notation:
  - Preference: $A \succ L_1$
  - Indifference: $L_2 \sim B$

Our Example

- L1 = [100% $1Billion, 0 % 0]
- L2 = [50% 0 $, 50% $10 Billion]

- Most people prefer L1 over L2.

How do we know our preferences are “rational”?

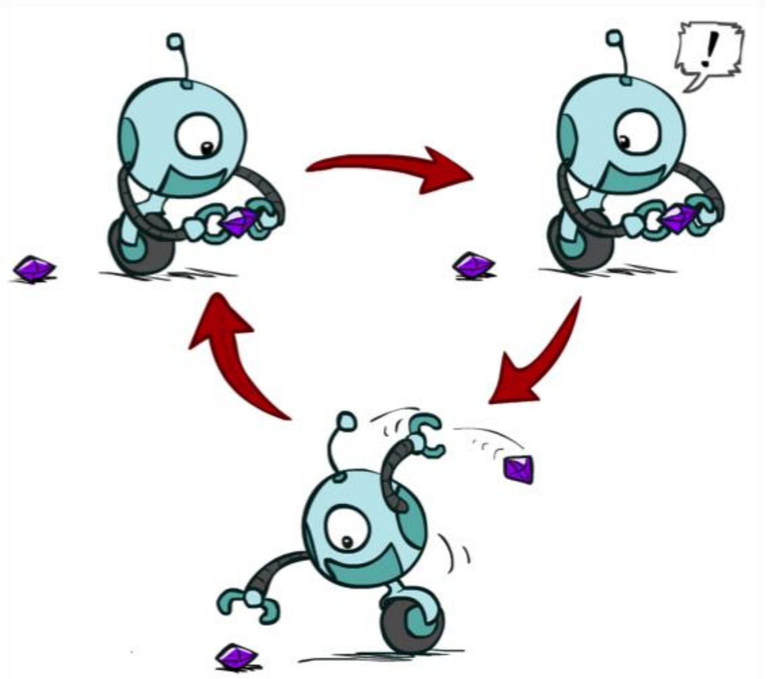
RATIONAL PREFERENCES

Rational Preferences

- We want some constraints on preferences before we call them rational, such as:

Axiom of Transitivity: $(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$

- For example: an agent with intransitive preferences can be induced to give away all of its money
  - If B > C, then an agent with C would pay (say) 1 cent to get B
  - If A > B, then an agent with B would pay (say) 1 cent to get A
  - If C > A, then an agent with A would pay (say) 1 cent to get C



Rational Preferences

The Axioms of Rationality

Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

Continuity

$$A \succ B \succ C \Rightarrow \exists p\ [p, A;\ 1-p, C] \sim B$$

Substitutability

$$A \sim B \Rightarrow [p, A;\ 1-p, C] \sim [p, B; 1-p, C]$$

Monotonicity

$$A \succ B \Rightarrow$$
$$(p \geq q \Leftrightarrow [p, A;\ 1-p, B] \succeq [q, A;\ 1-q, B])$$

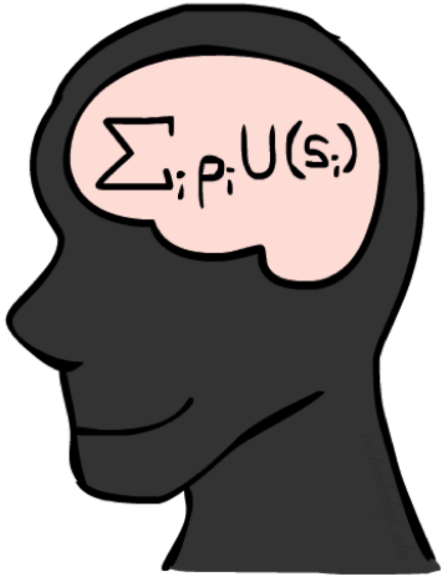
Theorem: Rational preferences imply behavior describable as maximization of expected utility

MEU Principle

- Theorem [Ramsey, 1931; von Neumann & Morgenstern, 1944]
  - Given any preferences satisfying these constraints, there exists a real-valued function U such that:

$$U(A) \geq U(B) \;\Leftrightarrow\; A \succeq B$$

$$U([p_1, S_1;\; \ldots \;;\; p_n, S_n]) = \sum_i p_i U(S_i)$$



  - I.e. values assigned by U preserve preferences of both prizes and lotteries!

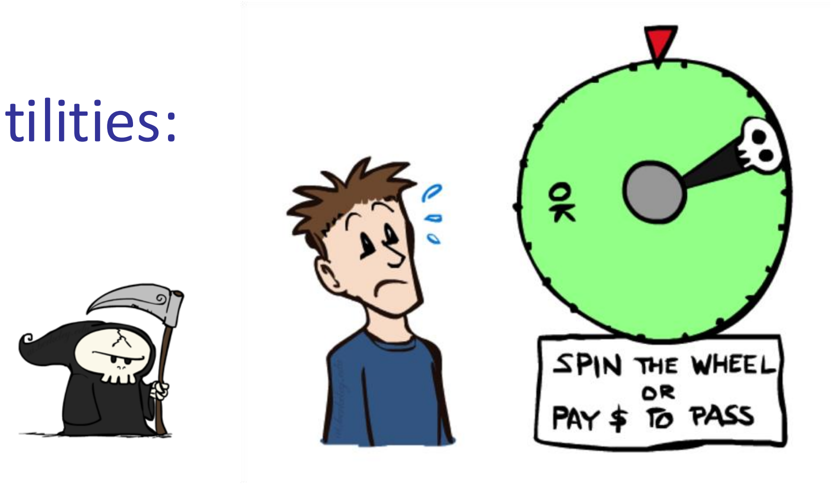
- Maximum expected utility (MEU) principle:
  - Choose the action that maximizes expected utility
  - Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table for perfect tic-tac-toe, a reflex vacuum cleaner

How do humans behave when thinking about utilities?

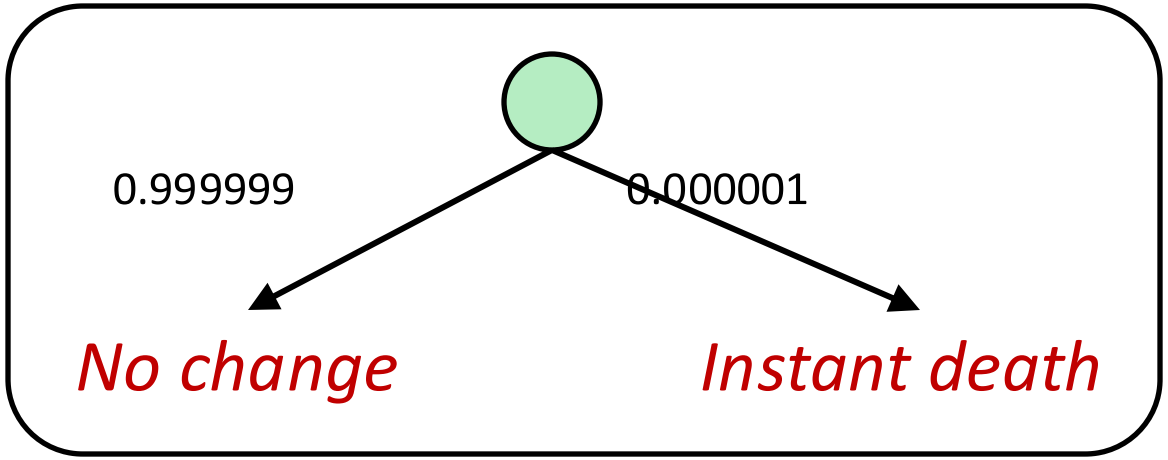
HUMAN UTILITIES

Human Utilities

- Utilities map states to real numbers. Which numbers?
- Standard approach to assessment (elicitation) of human utilities:
  - Compare a prize A to a standard lottery L_p between
    - "best possible prize" u_+ with probability p
    - "worst possible catastrophe" u_- with probability 1-p
  - Adjust lottery probability p until indifference: $A \sim L_p$
  - Resulting p is a utility in [0,1]

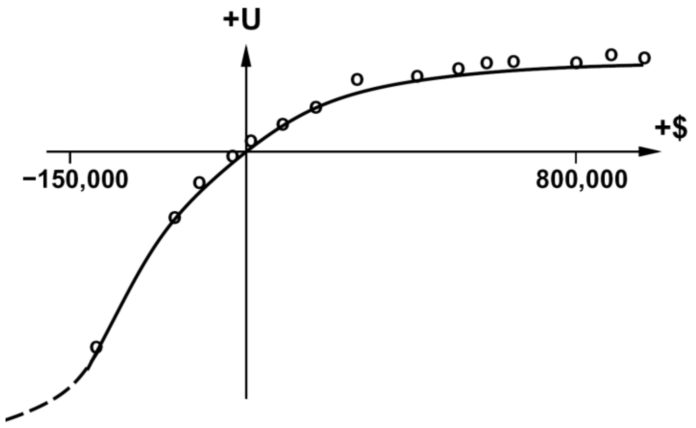


Pay $30 ~



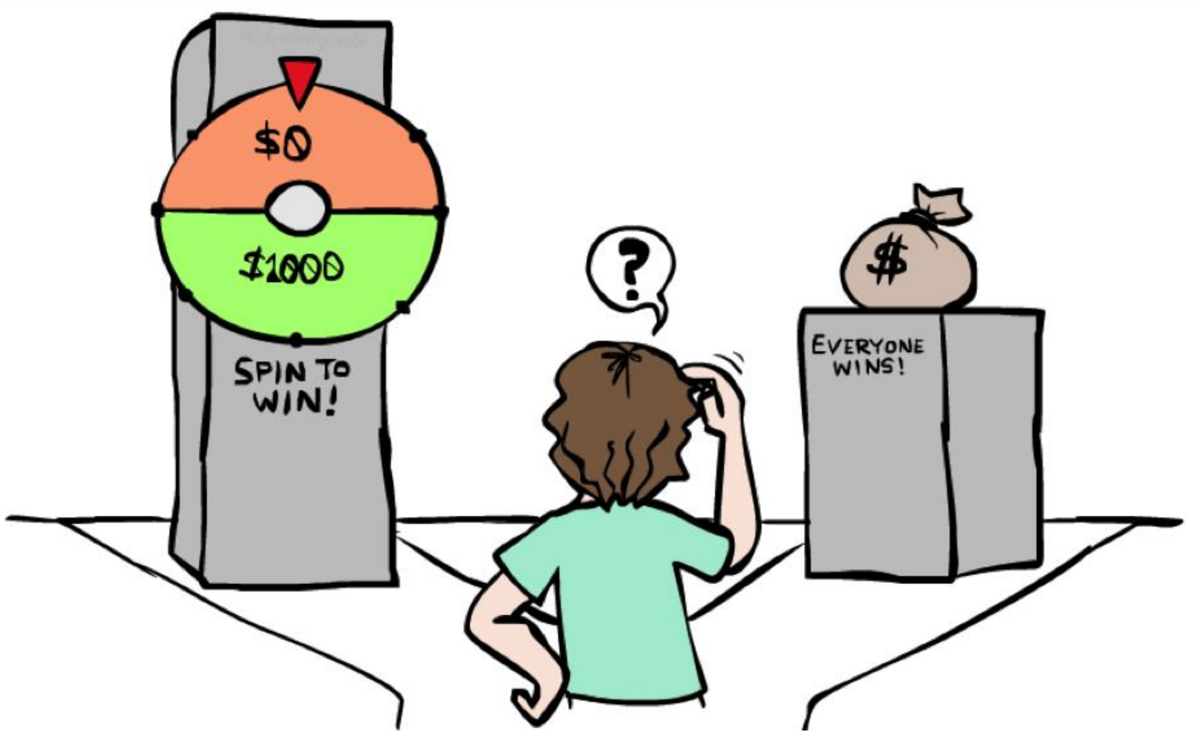
Money

- Money <u>does not</u> behave as a utility function, but we can talk about the utility of having money (or being in debt)
- Given a lottery L = [p, $X; (1-p), $Y]
  - The expected monetary value EMV(L) is p*X + (1-p)*Y
  - U(L) = p*U($X) + (1-p)*U($Y)
  - Typically, U(L) < U(EMV(L))
  - In this sense, people are risk-averse
  - When deep in debt, people are risk-prone



Example: Insurance

- Consider the lottery [0.5, $1000; 0.5, $0]
  - What is its expected monetary value? ($500)
  - What is its certainty equivalent?
    - Monetary value acceptable in lieu of lottery
    - $400 for most people
  - Difference of $100 is the insurance premium
    - There's an insurance industry because people will pay to reduce their risk
    - If everyone were risk-neutral, no insurance needed!
  - It's win-win: you'd rather have the $400 and the insurance company would rather have the lottery (their utility curve is flat and they have many lotteries)



Not a Discrepancy!

- This is not a discrepancy! It merely conveys that for the person taking the survey:
- U($1000) ~ 2 * U($400)

Choose one

- Option 1: Win (with 50% probability) two billion dollars, or Option 2: Win (with 100% probability) one billion dollars
  - L1 = L[0.5, 0B; 0.5, 2B]
  - L2 = L[1.0, 1B; 0, 0$]
- For most people, U(L1) < U(L2), though EMV(L1) = EMV(L2)
- 0.5 U(2B) < U(1B)
- In fact, for most people:
  - U(L[0.9, 0B; 0.1, 100B]) < U(L[1.0, 1B; 0, 0$]), though EMV(L1) = 10B, EMV(L2) = 1B

Example: Human Rationality?

- Given
  - A: [0.8, $4k;   0.2, $0]
  - B: [1.0, $3k;   0.0, $0]

- Which one do you choose?

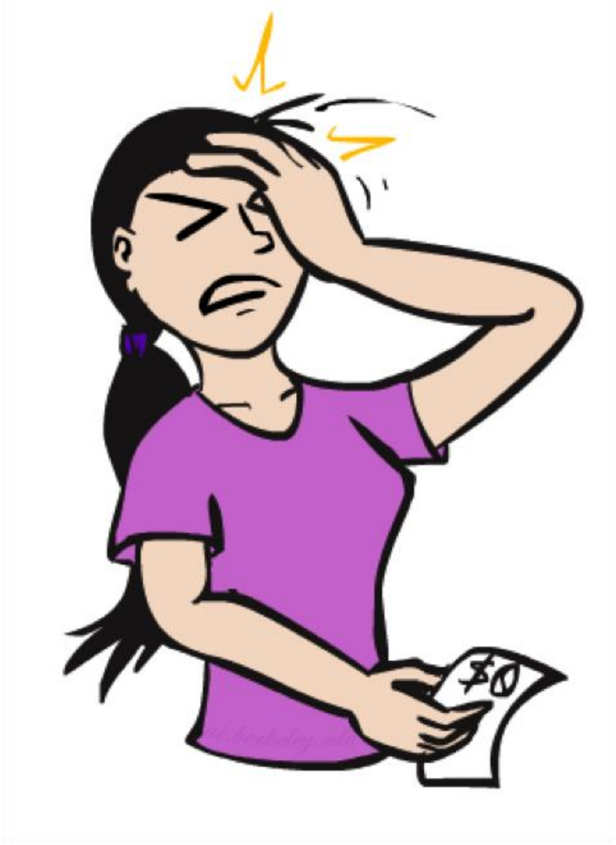
- Given
  - C: [0.2, $4k;   0.8, $0]
  - D: [0.25, $3k;   0.75, $0]

- Which one do you choose?

https://en.wikipedia.org/wiki/Allais_paradox

Example: Human Rationality?

- Famous example of Allais (1953)
  - A: [0.8, $4k;   0.2, $0]
  - B: [1.0, $3k;   0.0, $0]
  - C: [0.002, $4k;   0.8, $0]
  - D: [0.0025, $3k;   0.75, $0]
- Most people prefer B > A, C > D
- But if U($0) = 0, then
  - B > A $\Rightarrow$ U($3k) > 0.8 U($4k)
  - C > D $\Rightarrow$ 0.8 U($4k) > U($3k)



Probability & Belief State

- Objectivist / frequentist answer:
  - Averages over repeated *experiments*
  - E.g., empirically estimating P(rain) from historical observation
  - Assertion about how future experiments will go (in the limit)
  - New evidence changes the *reference class*
  - Makes one think of *inherently random* events, like rolling dice

- Subjectivist / Bayesian answer:
  - Degrees of belief about unobserved variables
  - E.g. an agent’s belief that it’s raining, given the temperature
  - E.g. pacman’s belief that the ghost will turn left, given the state
  - Often *learn* probabilities from past experiences (more later)
  - New evidence *updates beliefs* (more later)

Practical Perspective: Utility Scales

- Normalized utilities: $u_+ = 1.0$, $u_- = 0.0$
- Micromorts: one-millionth chance of death, useful for paying to reduce product risks, etc.
- https://www.stubbornmule.net/2010/12/micromorts/

- QALYs: quality-adjusted life years, useful for medical decisions involving substantial risk
- Note: behavior is invariant under positive linear transformation
- With deterministic prizes only (no lottery choices), only ordinal utility can be determined, i.e., total order on prizes

$$U'(x) = k_1 U(x) + k_2 \quad \text{where } k_1 > 0$$

Uncertainty Everywhere

- Not just for games of chance!
  - I'm sick: will I sneeze this minute?
  - Email contains "FREE!": is it spam?
  - Tooth hurts: have cavity?
  - 60 min enough to get to the airport?
  - Robot rotated wheel three times, how far did it advance?
  - Safe to cross street? (Look both ways!)

- Sources of uncertainty in random variables:
  - Inherently random process (dice, etc)
  - Insufficient or weak evidence
  - Ignorance of underlying processes
  - Unmodeled variables
  - The world's just noisy – it doesn't behave according to plan!

- Compare to *fuzzy logic*, which has *degrees of truth*, rather than just *degrees of belief*